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Note

On a Polynomial Inequality of Turán

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Let $P(x)=\prod_{k=1}^{n}(x-\alpha_k)$ be a polynomial of degree n. P. Turán proved the inequality (see [1])

$$\prod_{|\alpha_k|\geqslant 1}|\alpha_k|\leqslant 2^{n-1}\|P\|_{L^\infty[-1,1]}$$

(empty products should be taken to be 1). Although this inequality is exact, for example, for the Chebyshev polynomials, if there are several large α_k's, then it is not sharp. The aim of this note is to discuss a related inequality which is better if $P(x)$ has many large zeros.

Our inequality is a consequence of the formula

$$\prod_{k=1}^{n}|\alpha_k+\sqrt{\alpha_k^2-1}\,|=2^n\exp\left(\frac{1}{\pi}\int_{-1}^{1}\frac{\log|P_n(t)|}{\sqrt{1-t^2}}\,dt\right) \tag{1}$$

where that branch of the square root is taken which is positive on the positive real half-line. In fact, this formula is an immediate consequence of the well known identity

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\log|\alpha-t|}{\sqrt{1-t^2}}\,dt=\log|\alpha+\sqrt{\alpha^2-1}\,|-\log 2.$$

Since $|\alpha+\sqrt{\alpha^2-1}\,|\geqslant 1$ for all $\alpha\in\mathbb{C}$ and $|\alpha+\sqrt{\alpha^2-1}\,|\geqslant(|\alpha|+\sqrt{|\alpha|^2-1})$ when $|\alpha|\geqslant 1$, with $w(x)=1/\pi\sqrt{1-x^2}$ we immediately obtain from (1) and the inequality between the geometric and power means that, for all $0<p\leqslant\infty$,

$$\prod_{|\alpha_k|\geqslant 1}(|\alpha_k|+\sqrt{|\alpha_k|^2-1})\leqslant 2^n\|P_n\|_{L^p(w)}. \tag{2}$$

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In particular, for $L \geqslant 1$

$$\prod_{|\alpha_k| \geqslant L} |\alpha_k| (1+\sqrt{1-L^{-2}}) \leqslant 2^n \|P_n\|_{L^p(w)}. \tag{3}$$

When $L=1$ and $p=\infty$ this is weaker by a factor of 1/2 than Turán's inequality, but (3) has advantages: instead of the supremum norm we can use weighted L^p-norms and the size of the large zeros is strongly taken into account. We also note that (2) is sharp in a certain sense, it would not be true, e.g., with 2^{n-1} instead of 2^n (consider the polynomials with zeros at L-times the nth roots of unity).

For the zeros on the imaginary axis formula (1) gives

$$\prod_{\substack{|\alpha_k| \geqslant 1 \\ \Re \alpha_k = 0}} (|\alpha_k| + \sqrt{|\alpha_k|^2+1}) \leqslant 2^n \|P_n\|_{L^p(w)}.$$

The above argument has the additional advantage that it can be used with other weights than w and also the interval $[-1, 1]$ can be replaced by other curves. Consider, e.g., the weight $v(x) = (2/\pi)\sqrt{1-\alpha^2}$ on $[-1, 1]$. Simple calculation shows that its potential is given by

$$\frac{2}{\pi}\int_{-1}^{1} \log|\alpha - t| \sqrt{1-t^2}\, dt = \log|\alpha + \sqrt{\alpha^2-1}| + \Re \frac{\alpha}{\alpha+\sqrt{\alpha^2-1}} - \frac{1}{2} - \log 2.$$

Here $\Re(\alpha/(\alpha+\sqrt{\alpha^2-1})) \geqslant \sqrt{2}-1$ if $|\alpha| \geqslant 1$ and for $\alpha \in [-1, 1]$ we have $\Re(\alpha/(\alpha+\sqrt{\alpha^2-1})) = \alpha^2 \geqslant 0$, hence by the maximum modulus principle for harmonic functions we can conclude that $\Re(\alpha/(\alpha+\sqrt{\alpha^2-1})) \geqslant 0$. Thus we get similarly as above

$$\prod_{|\alpha_k| \geqslant 1} (|\alpha_k| + \sqrt{|\alpha_k|^2-1})\, e^{\sqrt{2}-1} \leqslant 2^n e^{n/2} \|P_n\|_{L^p(v)}.$$

The larger constant factor in this inequality (cf. (2)) is compensated by the smaller weight v.

Reference

1. P. Turán, On an inequality of Čebyšev, *Ann. Univ. Sci. Eötvös, Budapest, Sect. Math.* **11** (1968), 15–16.

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